# Headsolving competition

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October 2023

**Acknowledgements**
Thank you to my friend Kevin for testing these out.
**Qualifiers (8 rounds): There are 5 Questions in front of both contestants. If one gets one wrong the point goes to the opponent. Whoever answers the most wins.**

1. Integrate
$$\int_0^{\frac{\pi}{2}} \cos(x)\sin(x)\,\mathrm{d}x$$
2. How many ways can you arrange 1 blue book, 3 red books, and 5 yellow books
3. Factorise $x^3 + 2x^2 + x + 2$ into (not necessarily linear) real components
4. Solve $e^{2x} - 2e^x - 15 = 0$
5. What is the inverse of $f(x) = \ln\left(\frac{2-x}{2+x}\right)$?
6. What is the prime factorisation of 1113?
7. Simplify $\tan(\arccos(x))$
8. What is the product of the 5th fibonacci number (starting with "1,1..."), 5th prime and 5th triangle number?
9. Express 314 as the sum of two squares
10. What are the interior angles of a regular dodecagon?
11. On a number line, if I start at 0 and move to 1 at 0.2 units per second, and you move from 1 to 0 at 0.1 units per second, how long will it take for us to pass by each other?
12. Round 1653 to the nearest multiple of 7.
13. Factorise $x^3 + 3x^2 - 28x - 60$ given $(x - 6)$ is a factor.
14. Expand $(x + 2)(x - 5)(x + 1)(x - 7)$
15. What is $\sqrt{7}$ to 1 decimal place?
16. What is the next term of the quadratic sequence starting with 5, 10, 19...
17. What is the last digit of $2^{284}$?
18. What is the horizontal asymptote of $f(x) = \frac{x-5}{7-x}$?
19. What is the coefficient (as a number) of $x^7$ in the Maclaurin expansion for $\sin(x)$?
20. Convert 23 to base 4.
21. How many primes are between 70 and 90?

22. Evaluate:

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} \left(\frac{1}{4}\right)^{n}$$

23. What is the highest common factor between 161 and 299

24. At what $x$ does $f(x) = \frac{\ln(x)}{x}$ reach its maximum?

25. Simplify $\sqrt{3 + 2\sqrt{2}}$

26. What is $\binom{6}{4}$?

27. What is the second number that is both a square and a triangle number?

28. How many 0s does 50! end in?

29. What are the first 2 decimal digits in the decimal expansion of $\frac{1}{7}$?

30. What is 0.027027027... as a simplified fraction?

31. What is $\arctan\left(\frac{1}{\sqrt{3}}\right) + \arctan\left(\sqrt{3}\right)$

32. What 3 positive cubes sum to 134?

33. Simplify $\sqrt{x + \sqrt{x + \sqrt{x + \ldots}}}$. Ignore issues of convergence.

34. How many equilateral triangles side length 1 does it take to tile the surface of a regular tetrahedron side length 2.

35. What is the sum of the interior angles of a nonagon?

36. What is $\cos\left(\frac{7\pi}{12}\right)$ in radicals?

37. Sum all natural numbers up to 19.

38. What is the area of a triangle with side lengths 7, 24, and 25?

39. I have a unit square and connect the midpoints to form a new square. I connect the midpoints of that square to form another. What is the area of the final square?

40. What is the largest number of unit equilateral triangles that can tile a parallelogram with angles 60° and 120° with parallel sides of length 4 and a slope of 3.

**Quarter-finals (4 rounds): You both see 2 questions with infinite guesses. Each wrong guess adds on 30 seconds to a timer. If you answer one question, you stop, and the opponent has to answer the other within that time (taking the difference of the time they generate with their mistakes). If they do, you lose.**

1. Give a 2 digit number that can be written as the sum of 2 distinct positive squares in 2 ways.
2. How many ways can you tile a 4 by 4 square grid with only L shaped tetrominoes [4 connected squares (edge to edge) in an L shape] (you can flip them and the board is fixed in orientation).
3. By drawing lines between integer coordinates, on a 3×3 grid, how many ways can I draw a square?
4. How many integer solutions does $x^2 - 24 = y^2$ have?
5. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{y}{x}$?
6. What is the equation, in the form $y = mx + c$ of tangent to the curve of $\ln(\ln(x))$ at $x = e^e$?
7. What is the first power of 2 ending in 68?
8. A currency has coin values of 1, 3, 6, and 10. What is the smallest number of coins required to get 85p?

**Semi-finals (2 rounds): You have 3 questions, with 1 guess per question. Whoever answers any question first correctly, or who;ever's opponent loses all their guesses, wins!**

1. Given $x_0 = 2$, $x_{n+1} = 2x_n^2$, find $x_n$.
2. Find $\sum_{n=0}^{\infty} n2^{-n}$.
3. A unit equilateral triangle is sitting on its base. How high up should I make a horizontal cut to split it into two pieces of equal area?
4. If I roll a fair 6-sided dice repeatedly. What is the chance of rolling 2 sixes before rolling any 1?.
5. What is the general solution to the SODE $f''(x) + 3f'(x) + 2f(x) = \cosh(x)$
6. Evaluate $\sum_{n=2}^{\infty} \frac{n^2 x^{n-1}}{n!}$

**Finals: 3 Questions and 3 guesses. If you use up all your guesses on 1 question it goes to your opponent; whoever answers the most questions wins!**

1. Find a divergent bounded sequence $a_n$ (closed form (i.e no integrals or sums)) such that $a_n - a_{n-1} \to 0$
2. In the domain $[-1, 1]$, what is the maximum of $f(x) = |3|x| + x + |4|x| - 1| - 4|$.
3. How many triangles can you draw with vertices at integer coordinates on $[1, 4]^2$?